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Discrete Wigner function for finite-dimensional systems

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Abstract. A phase-space approach to finite-dimensional systems is developed from basic principles. For a system describable by a Hilbert space of dimension d we define a one-to-one correspondence between operators and functions on a discrete and finite phase space with d^2 points valid for any dimension d. The properties fulfilled by this correspondence and its uniqueness are examined. This formalism is applied to the number difference and phase difference of a two-mode field. This case is compared with the marginal distribution for these variables arising from a two-mode Wigner function for number and phase.

1. Introduction

Soon after the beginning of quantum mechanics there appeared its first phase-space formulation, the Wigner–Weyl formalism or Wigner function [1], fully equivalent to standard Hilbert space quantum mechanics. States and observables are replaced by functions on the classical phase space and expected values are computed, as in classical statistical physics, by averaging over the phase space. This formalism, like other generalizations [2], applies to systems describable by Cartesian conjugate variables, such as position and linear momentum, which are unbounded continuous degrees of freedom. This covers a great variety of situations including, for example, field modes where quadratures play the role of position and momentum [2].

Since then, this formalism has been translated to other different situations. However, phase-space formalisms based on conjugate variables other than Cartesian ones (or their linear combinations) are not straightforward from the original formulation and must be reconstructed from first principles. This is the case of Wigner functions involving variables such as action and phase [3, 4] or angle and angular momentum [5, 6].

Here we will focus on a Wigner–Weyl formalism for systems described by finite-dimensional Hilbert spaces [6–9]. Among them, spin systems are a proper example although it can be applied to other situations [6]. As occurs for Cartesian variables [2], there are also other phase-space methods for finite-dimensional systems as can be seen in [10].

A first and necessary ingredient is a suitable definition of phase space. At this stage two main possibilities emerge. We can use a bounded and continuous space (the sphere, for example [7]) or a discrete and finite set of points [6, 8, 9]. There is also the possibility

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of discreteness for one variable and continuity for the other. Their difference is a matter of the properties attached to the correspondence between operators and functions.

Here we will follow the possibility of a discrete and finite phase space. For a system described by a Hilbert space of dimension d we will consider a set of d^2 points as its phase space. This phase space will be formed by the spectrum of two conjugate variables. A Wigner–Weyl formalism based directly on a phase space with d^2 points is known in the case of odd dimension d. For even dimension, previous approaches resort to decompose the space into spaces of odd dimension according to the prime factorization of d [8]. On the other hand, there are formalisms valid simultaneously for even and odd dimension, but based on an enlarged phase space containing more than d^2 points [6, 9].

We will construct the most general correspondence between operators and functions delimiting it by imposing desirable properties. These properties are briefly listed in the appendix. This procedure has the advantage that besides leading us to the solution we are looking for (in the case when it exists) it also gives simultaneously whether it is unique or not in addition to a set of compatible properties. Previous solutions for odd d will emerge and their uniqueness and properties will be examined.

For the sake of illustration the system is assumed to be describing a spin. This choice is merely a matter of convenience and the solution is valid for every finite-dimensional system after properly renaming the variables involved. As a further example, in section 3 we will consider a phase-space description of number and phase difference for a two-mode field. This is possible because these are variables are compatible with the total photon number and all subspaces with given total photon number have finite dimension.

This example offers an interesting possibility. Since there are phase-space formalisms in terms of number and phase for a one-mode field [3, 4], a phase-space description for number and phase difference can be derived form a two-mode Wigner function by the corresponding marginal. This provides a different procedure for the definition of a Wigner–Weyl correspondence for finite-dimensional systems which will be examined in section 4.

2. Discrete Wigner function

The Wigner function, or Wigner–Weyl correspondence, is a rule associating linear operators A acting on the Hilbert space of the system with functions W_A on the corresponding phase space. Its purpose is to obtain a system description fully equivalent to standard quantum mechanics having a formal similarity with classical statistical mechanics as far as possible.

For simplicity, we will consider that the finite-dimensional space describes an angular momentum $\boldsymbol{j}$. A suitable phase space is formed by two conjugate observables on the Hilbert space H_j with dimension $d = 2j + 1$, $j = 1/2, 1, \ldots$. As first variable we choose one of the Cartesian components of $\boldsymbol{j}$, j_z for instance. Its eigenvectors and eigenvalues are (in units $\hbar = 1$)

$$j_z|j, m\rangle = m|j, m\rangle \qquad m = -j, -j+1, \ldots, j-1, j. \tag{2.1}$$

Its canonically conjugate variable will be the azimuthal angle $\phi = \arg(j_x + \mathrm{i}j_y)$ described in quantum terms by the unitary operator E exponential in ϕ [11] with eigenvectors and eigenvalues

$$E|j, \phi_s\rangle = \mathrm{e}^{\mathrm{i}\phi_s}|j, \phi_s\rangle \tag{2.2}$$

where

$$|j, \phi_s\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} \mathrm{e}^{-\mathrm{i}m\phi_s}|j, m\rangle \tag{2.3}$$

and

$$\phi_s = \frac{2\pi}{2j+1}s \qquad s = -j, -j+1, \ldots, j-1, j. \tag{2.4}$$

For the sake of symmetry between j_z and ϕ the following unitary operator F can be defined

$$F = \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))j_z}. \tag{2.5}$$

The operators E and F satisfy [11, 12]

$$E^k F^\ell = \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))k\ell} F^\ell E^k \tag{2.6}$$

which expresses a proper commutation relation between j_z and ϕ in the Weyl form, so they can be considered as conjugate variables. A discrete and finite phase space can be formed by the set of $(2j+1)^2$ points (m, ϕ_s) which we will denote by (m, s) for simplicity.

In the appendix we recall properties for the Wigner function dictated by the equivalence with the Hilbert space description, its statistical interpretation and proper behaviour under transformations [1, 5, 7]. Next we study how these properties delimit the correspondence.

By linearity, we may assume that the correspondence between functions $W_A(m, s)$ and operators A is performed by a family of $(2j+1)^2$ operators $\Delta(m, s)$ (operator kernel or phase points operators) in the form

$$W_A(m, s) = \mathrm{tr}[A\Delta(m, s)]. \tag{2.7}$$

Next we determine the requirements that $\Delta(m, s)$ should fulfill in order to guarantee the properties of the Wigner function listed in the appendix.

Property (A.1) gives that if $A \neq 0$ then $W_A \neq 0$ so if $A \neq B$ then $W_A \neq W_B$ and different operators must have different Wigner functions. By suitably applying (A.1) and (2.7) it can be shown that it is possible to invert (2.7) in the form

$$A = \frac{1}{2j+1} \sum_{m,s=-j}^{j} W_A(m, s)\Delta(m, s). \tag{2.8}$$

According to this inverse relation, the family $\Delta(m, s)$ must be a basis of operators and, since its number is $(2j+1)^2$, they must be linearly independent. This means that different functions will give different operators by relation (2.8). Using (2.8) with $A = \Delta(m', s')$ it follows from their linear independence that

$$\mathrm{tr}[\Delta(m', s')\Delta(m, s)] = (2j+1)\delta_{m,m'}\delta_{s,s'}. \tag{2.9}$$

Vice versa, this orthogonality relation accounts for the inversion formula (2.8) and the tracial property (A.1) simultaneously. As a bonus we have obtained that the correspondence $A \leftrightarrow W_A$ is one to one on both directions as a consequence of the number of phase-space points. In other words, if a one-to-one correspondence is required the number of phase-space points must be $(2j+1)^2$.

The reality condition (A.2) means that the operator kernel must be Hermitian

$$\Delta^\dagger(m, s) = \Delta(m, s). \tag{2.10}$$

Proper marginal distributions (A.3) lead to

$$\begin{aligned} \frac{1}{2j+1} \sum_{s=-j}^{j} \Delta(m, s) &= |j, m\rangle\langle j, m| \\ \frac{1}{2j+1} \sum_{m=-j}^{j} \Delta(m, s) &= |j, \phi_s\rangle\langle j, \phi_s|. \end{aligned} \tag{2.11}$$

Suitable transformation under translations (A.5) gives the condition

$$F^{\ell'} E^{\dagger k'} \Delta([[m+k']], [[s+\ell']]) E^{k'} F^{\dagger \ell'} = \Delta(m,s) \tag{2.12}$$

where $[[m]]$ is equal to m modulus $2j+1$, being $[[m]] \in [-j, j]$. The parity transformation (A.7) and Fourier transformation (A.9) lead to

$$P\Delta(-m,-s)P = \Delta(m,s) \tag{2.13}$$

and

$$T^{\dagger}\Delta(-s,m)T = \Delta(m,s). \tag{2.14}$$

Finally the conjugations (A.10) and (A.11) impose that

$$\begin{aligned} \langle j, m' | \Delta(m,-s) | j, m'' \rangle^* &= \langle j, m' | \Delta(m,s) | j, m'' \rangle \\ \langle j, \phi_{s'} | \Delta(-m,s) | j, \phi_{s''} \rangle^* &= \langle j, \phi_{s'} | \Delta(m,s) | j, \phi_{s''} \rangle. \end{aligned} \tag{2.15}$$

At this stage it is not yet granted that all conditions can be satisfied simultaneously nor whether they define the correspondence uniquely. These points will be answered after finding solutions for these equations. We will follow a constructive method starting with the most general operator kernel then imposing the properties (2.9)–(2.15).

To this end, we express $\Delta(m,s)$ in an operator basis. The $E^k F^\ell$ basis is a convenient one so we will consider

$$\Delta(m,s) = \sum_{k,\ell} \Delta_{k,\ell}(m,s) E^k F^\ell \tag{2.16}$$

where $\Delta_{k,\ell}(m,s)$ are undetermined coefficients depending on (m,s). In this expression the range of variation of k and ℓ is

$$\begin{aligned} k &= k_0, k_0+1, \ldots, k_0+2j = \{k_0\} \\ \ell &= \ell_0, \ell_0+1, \ldots, \ell_0+2j = \{\ell_0\} \end{aligned} \tag{2.17}$$

where k_0 and ℓ_0 are arbitrary integers. These operators $E^k F^\ell$ are orthonormal with respect to the trace product

$$\mathrm{tr}[(E^{k'}F^{\ell'})^\dagger E^k F^\ell] = \mathrm{tr}(F^{\dagger\ell'} E^{\dagger k'} E^k F^\ell) = (2j+1)\delta_{k',k}\delta_{\ell',\ell} \tag{2.18}$$

where $k, k' \in \{k_0\}$ and $\ell, \ell' \in \{\ell_0\}$. They are linearly independent and complete because its number matches the dimension of the algebra of operators acting on H_j.

We start with translations (2.12) which give the equality

$$\sum_{k,\ell} \Delta_{k,\ell}([[m+k']], [[s+\ell']])\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))(k\ell'+k'\ell)} E^k F^\ell = \sum_{k,\ell} \Delta_{k,\ell}(m,s) E^k F^\ell \tag{2.19}$$

leading to

$$\Delta_{k,\ell}(m,s) = \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} \Delta_{k,\ell} \tag{2.20}$$

where $\Delta_{k,\ell}$ no longer depend on m and s.

To impose the orthogonality (2.9) we assume momentarily the reality (2.10) and with the help of (2.18) we arrive at

$$\sum_{k,\ell} \Delta^*_{k,\ell}(m',s')\Delta_{k,\ell}(m,s) = \delta_{s,s'}\delta_{m,m'}. \tag{2.21}$$

Using (2.20) we have

$$\Delta_{k,\ell} = \frac{1}{2j+1}\,\mathrm{e}^{\mathrm{i}\gamma_{k,\ell}} \tag{2.22}$$

where $\gamma_{k,\ell}$ are undetermined phases.

Next we consider the reality condition (2.10) which can be written, using the expressions (2.20), (2.22) and the commutation relation (2.6), as

$$\sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))(ks+\ell m)}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))k\ell} E^{\dagger k} F^{\dagger \ell} = \sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} E^{k} F^{\ell}. \quad (2.23)$$

We can rearrange the sum on the left-hand side in the following form. Similarly as in (2.12) and (A.5) $[[-k]]$ denotes the integer equal to $-k$ modulus $2j+1$, being $[[-k]] \in \{k_0\}$. To each k, ℓ there corresponds one and only one $[[-k]], [[-\ell]]$. This allows us to rearrange the left-hand side of (2.23) in the form

$$\begin{aligned} &\sum_{k,\ell} \mathrm{e}^{-\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))([[-k]]s+[[-\ell]]m)}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))[[-k]][[-\ell]]} E^{[[-k]]} F^{[[-\ell]]} \\ &\qquad = \sum_{k,\ell} \mathrm{e}^{-\mathrm{i}\gamma_{[[-k]],[[-\ell]]}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))k\ell} E^{k} F^{\ell} \end{aligned} \quad (2.24)$$

where $k = [[-[[-k]]]]$, $\ell = [[-[[-\ell]]]]$ have been used and it should be taken into account that $E^{2j+1} = F^{2j+1} = (-1)^{2j}$.

Using this form in (2.23) we obtain a relation between the $\gamma_{k,\ell}$ phases

$$\gamma_{k,\ell} + \gamma_{[[-k]],[[-\ell]]} = \frac{2\pi}{2j+1} k\ell. \quad (2.25)$$

We continue with the marginal for ϕ in (2.11) which gives

$$\frac{1}{(2j+1)^2} \sum_{m=-j}^{j} \sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} E^{k} F^{\ell} = |j, \phi_s\rangle\langle j, \phi_s|. \quad (2.26)$$

The m sum is

$$\frac{1}{2j+1} \sum_{m=-j}^{j} \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))\ell m} F^{\ell} = \mathrm{e}^{i(2\pi/(2j+1))j\ell} F^{\ell} \delta_{\ell,[[0]]} = \delta_{\ell,[[0]]}. \quad (2.27)$$

This gives the equation to be fulfilled as

$$\frac{1}{2j+1} \sum_{k} \mathrm{e}^{-\mathrm{i}\gamma_{k,[[0]]}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))ks} E^{k} = |j, \phi_s\rangle\langle j, \phi_s| \quad (2.28)$$

giving $\gamma_{k,[[0]]} = 0$. Similarly, the marginal for j_z gives the condition $\gamma_{[[0]],\ell} = 0$.

With respect to the parity transformation (2.13) we have the equality

$$\sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} E^{\dagger k} F^{\dagger \ell} = \sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} E^{k} F^{\ell}. \quad (2.29)$$

We can follow the same steps as in the procedure from (2.23) to (2.25) leading to the condition

$$\gamma_{[[-k]],[[-\ell]]} = \gamma_{k,\ell}. \quad (2.30)$$

The Fourier transformation (2.14) gives the equality

$$\sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(km-\ell s)}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))k\ell} E^{\dagger \ell} F^{k} = \sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} E^{k} F^{\ell}. \quad (2.31)$$

We can rearrange the left-hand sum as in (2.23) to arrive at

$$\gamma_{k,\ell} - \gamma_{\ell,[[-k]]} = \frac{2\pi}{2j+1} k\ell. \quad (2.32)$$

Finally, we consider conjugation (2.15) in the phase basis for instance, which gives the equality

$$\sum_{k,\ell} \mathrm{e}^{-\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))(ks-\ell m)} \langle j, \phi_{s'}|E^k F^\ell|j, \phi_{s''}\rangle^* = \sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} \langle j, \phi_{s'}|E^k F^\ell|j, \phi_{s''}\rangle. \tag{2.33}$$

The matrix element we need in this equation is

$$\langle j, \phi_{s'}|E^k F^\ell|j, \phi_{s''}\rangle = \mathrm{e}^{\mathrm{i}k\phi_{s'}} \langle j, \phi_{s'}|j, \phi_{s''-\ell}\rangle. \tag{2.34}$$

The scalar product on the right-hand side is proportional to the Kronecker delta $\delta_{\ell,[[s''-s']]}$, so the equality to be fulfilled is

$$\sum_k \mathrm{e}^{-\mathrm{i}\gamma_{k,[[s''-s']]}}\, \mathrm{e}^{\mathrm{i}(2\pi/(2j+1))k(s-s')} = \sum_k \mathrm{e}^{\mathrm{i}\gamma_{k,[[s''-s']]}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))k(s-s')} \tag{2.35}$$

which leads to

$$\gamma_{k,\ell} + \gamma_{[[-k]],\ell} = 0. \tag{2.36}$$

Similarly, the conjugation in the j_z basis gives

$$\gamma_{k,\ell} + \gamma_{k,[[-\ell]]} = 0. \tag{2.37}$$

For the sake of clarity we can summarize the conclusions obtained so far. The operator kernel must have the form

$$\Delta(m, s) = \frac{1}{2j+1} \sum_{k,\ell} \mathrm{e}^{\mathrm{i}\gamma_{k,\ell}}\, \mathrm{e}^{-\mathrm{i}(2\pi/(2j+1))(ks+\ell m)} E^k F^\ell \tag{2.38}$$

where the phases $\gamma_{k,\ell}$ must satisfy the equations

$$\gamma_{k,\ell} + \gamma_{[[-k]],[[-\ell]]} = \frac{2\pi}{2j+1} k\ell \qquad \text{reality} \tag{2.39a}$$

$$\gamma_{k,\ell} = \gamma_{[[-k]],[[-\ell]]} \qquad \text{inversion} \tag{2.39b}$$

$$\gamma_{k,[[0]]} = \gamma_{[[0]],\ell} = 0 \qquad \text{marginals} \tag{2.39c}$$

$$\gamma_{k,\ell} - \gamma_{\ell,[[-k]]} = \frac{2\pi}{2j+1} k\ell \qquad \text{Fourier} \tag{2.39d}$$

$$\gamma_{k,\ell} + \gamma_{[[-k]],\ell} = 0 \qquad \gamma_{k,\ell} + \gamma_{k,[[-\ell]]} = 0 \qquad \text{conjugation} \tag{2.39e}$$

where a modulus 2π for $\gamma_{k,\ell}$ must be understood throughout.

It can be easily seen that it is always possible to solve (2.39*a*)–(2.39*c*) simultaneously. This proves that it is possible to establish a one-to-one correspondence $A \leftrightarrow W_A$ guaranteeing the statistical interpretation of this phase-space formalism, its equivalence with Hilbert space quantum mechanics and basic transformation laws. No distinction has been made between even and odd dimension so no enlargement of the phase space is needed for the fulfillment of these basic properties.

The study of the possible fulfillment of conditions (2.39*d*) and (2.39*e*) explicitly involves the operations $[[-k]]$, $[[-\ell]]$ which have a special feature for even dimension as we shall see. For this reason it is convenient to split the joint procedure followed up to here.

2.1. Odd dimension

When $2j + 1$ is odd (j integer) we have a simple choice for the set of integers $\{k_0\}$ and $\{\ell_0\}$ which is $k_0, \ell_0 = -j$ so $k, \ell = -j, -j+1, \ldots, j$, and we have simply $[[-k]] = -k, [[-\ell]] = -\ell$ for all k, ℓ. The case with $j = 2$ is illustrated in figure 1. Conditions (2.39a) and (2.39b) lead to an expression for $\gamma_{k,\ell}$ in the form

$$\gamma_{k,\ell} = \frac{\pi}{2j+1} k\ell + \varepsilon_{k,\ell} \tag{2.40}$$

where $\varepsilon_{k,\ell}$ can be 0 and π only. Then we have $\varepsilon_{k,\ell} = \pm\varepsilon_{k,\ell}$ modulus 2π and the conditions (2.39) for $\gamma_{k,\ell}$ become equations for $\varepsilon_{k,\ell}$ that can be written in the form

$$\varepsilon_{-k,-\ell} = \varepsilon_{k,\ell} \tag{2.41a}$$

$$\varepsilon_{k,0} = \varepsilon_{0,\ell} = 0 \tag{2.41b}$$

$$\varepsilon_{\ell,-k,} = \varepsilon_{k,\ell} \tag{2.41c}$$

$$\varepsilon_{k,-\ell} = \varepsilon_{k,\ell} \qquad \varepsilon_{-k,\ell} = \varepsilon_{k,\ell}. \tag{2.41d}$$

These conditions mean that $\varepsilon_{k,\ell}$ takes the same value (0 or π) on those k, ℓ points connected by the corresponding transformations on the k, ℓ lattice; these are, inversion (2.41a), $\pi/2$ rotation (2.41c) and reflections (2.41d). We can find $\varepsilon_{k,\ell}$ values satisfying simultaneously all conditions (2.41) simply by assigning the same value 0 or π to the k, ℓ points connected with a given one by this set of operations, giving

$$\varepsilon_{k,\ell} = \varepsilon_{-k,-\ell} = \varepsilon_{-k,\ell} = \varepsilon_{k,-\ell} = \varepsilon_{\ell,k} = \varepsilon_{-\ell,k} = \varepsilon_{\ell,-k} = \varepsilon_{-\ell,-k} = 0, \pi \tag{2.42}$$

for each k, ℓ. By condition (2.41b) this value must be 0 if these points lie on the axes. Otherwise it can be 0 or π as well.

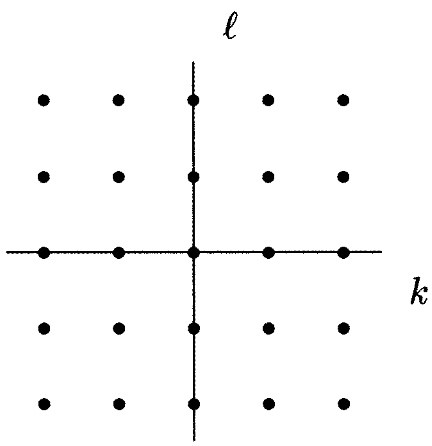

Figure 1. Illustration of the range of variation of k, ℓ for $j = 2$ when $k_0 = \ell_0 = -j = -2$.

The conclusion is that for odd dimension it is possible to define a Wigner–Weyl correspondence with all properties listed in the appendix in more than one way.

Next we study whether it is possible to express the operator kernel in other forms equivalent to (2.38). Since m and s are always integers, the operator kernel can be written in the form

$$\Delta(m, s) = D(m, s)\Delta(0, 0)D^{\dagger}(m, s) = E^m F^{\dagger s}\Delta(0, 0)F^s E^{\dagger m} \tag{2.43}$$

where $D(m, s)$ is in equation (A.4) and

$$\Delta(0,0) = \frac{1}{2j+1} \sum_{k,\ell} e^{i\varepsilon_{k,\ell}} e^{i(\pi/(2j+1))k\ell} E^k F^\ell. \tag{2.44}$$

It is known that for the Wigner function for position and linear momentum, the operator kernel is the displaced parity operator [2]. The question is whether there is a choice of $\varepsilon_{k,\ell}$ satisfying (2.41) such that $\Delta(0, 0)$ is the parity operator P in (A.6). To this end we will express P in the $E^k F^\ell$ basis

$$P = \sum_{k,\ell} P_{k,\ell} E^k F^\ell \qquad P_{k,\ell} = \frac{1}{2j+1} \mathrm{tr}(F^{\dagger\ell} E^{\dagger k} P). \tag{2.45}$$

The trace gives

$$P_{k,\ell} = \frac{1}{2j+1} \sum_{s=-j}^{j} e^{-i(2\pi/(2j+1))ks} \langle j, \phi_{-s} | j, \phi_{s+\ell} \rangle = \frac{1}{2j+1} e^{-i\pi k \mu_\ell} e^{i(\pi/(2j+1))k\ell} \tag{2.46}$$

where μ_ℓ is an integer such that $2s = -\ell + \mu_\ell(2j + 1)$ and it can be seen that for each value of ℓ this relation is fulfilled by one and only one integer $s \in [-j, j]$. Since $2j + 1$ is odd, μ_ℓ is odd (even) when ℓ is odd (even) and then

$$P = \frac{1}{2j+1} \sum_{k,\ell} e^{i\eta_{k,\ell}} e^{i(\pi/(2j+1))k\ell} E^k F^\ell \tag{2.47}$$

where $\eta_{k,\ell} = -\pi k \mu_\ell$ (modulus 2π) are given by

$$\eta_{k,\ell} = \begin{cases} 0 & \text{when } k \text{ and/or } \ell \text{ are even} \\ \pi & \text{when } k \text{ and } \ell \text{ are odd.} \end{cases} \tag{2.48}$$

We can see that $\eta_{k,\ell}$ satisfy (2.41) because these relations never mix k and ℓ odd with the other possibilities. If we take $\varepsilon_{k,\ell} = \eta_{k,\ell}$ as a particular solution of (2.41) we get $\Delta(0, 0) = P$ and the operator kernel becomes

$$\Delta(m, s) = D(m, s) P D^\dagger(m, s) = D(2m, 2s) P \tag{2.49}$$

which are formally the same expressions valid for the ordinary Wigner function [2]. The first one can be used to obtain the following expression in the $|j, \phi_s\rangle$ basis:

$$\Delta(m, s) = \sum_{r=-j}^{j} e^{2im\phi_r} |j, \phi_s + \phi_r\rangle\langle j, \phi_s - \phi_r|. \tag{2.50}$$

This shows that after the particular choice (2.48) for $\varepsilon_{k,\ell}$ we recover a Wigner–Weyl formalism valid only for odd dimension previously introduced [8]. We have found some of its properties and also that these forms (2.49) and (2.50) correspond to a particular solution for $\varepsilon_{k,\ell}$ in (2.41). This means that there are other admissible choices, all of them having the same set of properties we have examined.

2.2. Even dimension

Here again, in order to simplify the examination of conditions (2.39), we will make a definite choice for the ranges $\{k_0\}$ and $\{\ell_0\}$. When $2j + 1$ is even (j half integer) we have that $(2j - 1)/2$ is integer. We will take $k_0 = \ell_0 = -(2j - 1)/2$ which gives the following ranges of variation

$$k, \ell = -\frac{2j-1}{2}, -\frac{2j-1}{2} + 1, \ldots, \frac{2j-1}{2}, \frac{2j+1}{2} \tag{2.51}$$

which are illustrated in figure 2 for $j = 5/2$.

The special feature of even dimension is that there are always one $k \neq [[0]]$ and one $\ell \neq [[0]]$ such that $[[-k]] = k, [[-\ell]] = \ell$. This occurs for k, ℓ equal to $(2j+1)/2$. For the remainder of k, ℓ we have $[[-k]] = -k, [[-\ell]] = -\ell$ as in the case of odd dimension. In figure 2 we have singled out the point $k = \ell = (2j+1)/2$ as the point R_3, the set of k, ℓ points with k or ℓ equal to $(2j+1)/2$ as the set R_2 and the remainder as the set R_1. This is advantageous because the operations $[[-k]], [[-\ell]]$ and the conditions (2.39) preserve this splitting and do not relate elements from different sets.

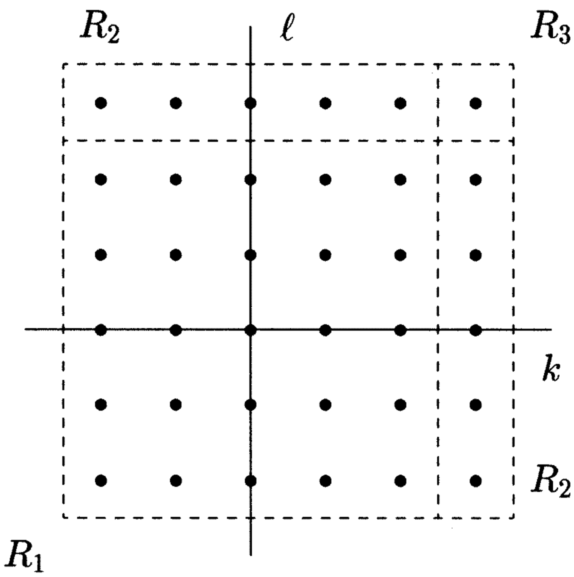

Figure 2. Range of variation of k, ℓ for $j = 5/2$ when $k_0 = \ell_0 = -(2j-1)/2 = -2$. The point $k = \ell = (2j+1)/2 = 3$ is the point R_3; points with k or ℓ equal to 3 form the set R_2 while the remainder of points which have k and ℓ different from 3 form the set R_1.

Within R_1 the conditions (2.39) behave exactly as in the case of odd dimension just studied, so all of them can be satisfied simultaneously in more than one way.

The situation is different within the other regions. In first place we can show that (2.39*e*) is not compatible with (2.39*a*) and (2.39*b*). For points R_2 and R_3 we have from (2.39*e*)

$$2\gamma_{(2j+1)/2,\ell} = 2\gamma_{k,(2j+1)/2} = 0 \tag{2.52}$$

whereas from (2.39*a*) and (2.39*b*)

$$2\gamma_{(2j+1)/2,\ell} = \pi\ell \qquad 2\gamma_{k,(2j+1)/2} = \pi k \tag{2.53}$$

so they cannot be satisfied simultaneously for k or ℓ odd. For any dimension there are always k and ℓ odd, so the conjugation property is excluded. It can be seen that (2.39*b*) can be derived form (2.39*e*), so conjugation is not compatible with reality which appears to be a more fundamental property.

Next we examine properties (2.39*a*)–(2.39*d*) by expressing $\gamma_{k,\ell}$ in the form (2.40). Concerning the point R_3 no conditions on $\varepsilon_{(2j+1)/2,(2j+1)/2}$ emerge from (2.39*a*)–(2.39*c*), while (2.39*d*) leads to

$$\pi\frac{2j+1}{2} = 0 \tag{2.54}$$

which can be satisfied only if $2j+1 = 4, 8, 12, \ldots$. Concerning points within R_2 we have the following conditions on $\varepsilon_{k,\ell}$

$$\varepsilon_{(2j+1)/2,-\ell} = \varepsilon_{(2j+1)/2,\ell} + \pi\ell \qquad \varepsilon_{-k,(2j+1)/2} = \varepsilon_{k,(2j+1)/2} + \pi k \tag{2.55a}$$
$$\varepsilon_{(2j+1)/2,0} = \varepsilon_{0,(2j+1)/2} = 0 \tag{2.55b}$$
$$\varepsilon_{\ell,(2j+1)/2} = \varepsilon_{(2j+1)/2,\ell} + \pi\ell \qquad \varepsilon_{k,(2j+1)/2} = \varepsilon_{(2j+1)/2,-k}. \tag{2.55c}$$

These conditions are compatible and can be solved simultaneously in more than one way. They mix $\varepsilon_{(2j+1)/2,\mu}, \varepsilon_{\mu,(2j+1)/2}, \varepsilon_{(2j+1)/2,-\mu}$ and $\varepsilon_{-\mu,(2j+1)/2}$. When μ is even, equations (2.55) give that all of them are equal

$$\varepsilon_{(2j+1)/2,\mu} = \varepsilon_{\mu,(2j+1)/2} = \varepsilon_{(2j+1)/2,-\mu} = \varepsilon_{-\mu,(2j+1)/2} = 0, \pi \tag{2.56}$$

being 0 if $\mu = 0$ and 0 or π are possible if $\mu \neq 0$. When μ is odd we have

$$\varepsilon_{\mu,(2j+1)/2} = \varepsilon_{(2j+1)/2,-\mu} = \pi + \varepsilon_{-\mu,(2j+1)/2} = \pi + \varepsilon_{(2j+1)/2,\mu} \tag{2.57}$$

and also in this case there is more than one solution.

Summarizing, for dimensions $2j+1 = 4, 8, 12, \ldots (j = 3/2, 7/2, 11/2, \ldots)$, it is possible to establish a one-to-one correspondence $A \leftrightarrow W_A$ with the properties of traciality, reality, proper marginals and suitable transformation laws under translations, parity and Fourier transforms, but not under conjugation. Concerning dimensions $2j+1 = 2, 6, 10, \ldots (j = 1/2, 5/2, 9/2, \ldots)$, we have all the properties except suitable transformation laws under Fourier transform and conjugation. In both cases the solution is not unique.

Finally, we could ask whether there is a particular set $\varepsilon_{k,\ell}$ providing expressions similar to (2.49) or (2.50). If it were possible to express the operator kernel as

$$\Delta(m,s) = U(m,s)PU^\dagger(m,s) \tag{2.58}$$

for some unitary operators $U(m,s)$ we would have

$$\mathrm{tr}\Delta(m,s) = \mathrm{tr}P = \sum_{m=-j}^{j} \langle j, -m|j, m\rangle = 0. \tag{2.59}$$

This would lead to $W_I(m,s) = 0$, where I is the identity, which is in contradiction with (A.1).

We could try the possibility of having the operator kernel in the form

$$\Delta(m,s) = U(m,s)P \tag{2.60}$$

for some unitary operators $U(m,s)$. Using the reality condition we would have

$$\Delta^2(m,s) = PU^\dagger(m,s)U(m,s)P = I \tag{2.61}$$

and the eigenvalues of $\Delta(m,s)$ should be ± 1. For even dimension, this is in contradiction with $\mathrm{tr}\Delta(m,s) = 1$ which follows from (A.3) and (2.8).

Therefore, (2.49) and (2.50) are excluded for even dimension. This does not mean that the Wigner function lacks desirable properties as we have demonstrated before.

3. Number difference and phase difference for a two-mode field

We have found the preceding section illustrative to identify the finite-dimensional space as describing an angular momentum. This choice is only a matter of convenience and the formalism developed can be applied to any other situation describable by a finite-dimensional space. In this section we will see that this formalism provides a phase-space description of number and phase difference of a two-mode field.

The Hilbert space of a two-mode field $H_1 \otimes H_2$ can be split as a direct sum of subspaces H_N with fixed total photon number N

$$H_1 \otimes H_2 = \sum_{N=0}^{\infty} H_N. \tag{3.1}$$

All subspaces H_N are finite-dimensional with dimension $N+1$, $N = 0, 1, \ldots$, and therefore isomorphic to the spaces H_j or preceding sections with $j = N/2$. We can perform this equivalence by means of the following relation between photon number states $|n_1, n_2\rangle$ and angular momentum states

$$|n_1 = N - n, n_2 = n\rangle \leftrightarrow |j = \tfrac{1}{2}N, m = \tfrac{1}{2}N - n\rangle \tag{3.2}$$

where $n = 0, 1, \ldots, N$. This relation allows us to rename the variables used in section 2 according to their meaning in this different context.

According to (3.2) we have that m is half of the number difference $(n_1 - n_2)/2$. The azimuthal angle ϕ will correspond to the phase difference which is the variable canonically conjugate to half of the photon number difference. The operator

$$E_{12} = \sum_{N=0}^{\infty} E_{12}^{(N)} \tag{3.3}$$

where $E_{12}^{(N)} \leftrightarrow E$ with $j = N/2$, is precisely a previously introduced unitary operator describing the exponential of the phase difference [13].

After all these identifications the phase-space formalism developed in section 2 provides in this context a phase-space description of the number difference and phase difference of a two-mode field. The correspondence between operators and functions is performed by the operator kernels $\Delta^{(N)}(m, \phi_s)$ defined as $\Delta^{(N)}(m, \phi_s) \leftrightarrow \Delta(m, \phi_s)$ with $j = N/2$.

In this case the associated phase space is discrete but no longer finite, being formed by all pairs (m, ϕ_s) for all possible values of N. In fact, these definitions provide a joint phase-space description of the total number, number difference and phase difference (N, m, ϕ_s) or, equivalently, photon numbers and phase difference (n_1, n_2, ϕ_s).

This is not a complete phase-space formalism for a two-mode field because number difference and phase difference are not a complete set of variables, even if we include the total number. It can be used only on situations involving these variables or any function of them, which means operators commuting with the total photon number. Accordingly, this phase-space description can be interpreted as a marginal quasidistribution of a complete Wigner function. Such a definition is out of the scope of this work since it would go beyond finite-dimensional spaces.

There are Wigner functions adapted to absolute number and phase [3, 4]. Their joint definition for a two-mode field provides another phase-space formalism for number and phase difference by means of the corresponding marginal distribution for these variables. We devote the next section to this procedure and its comparison with the one followed up to here.

4. Wigner function for absolute and difference number and phase variables

Wigner functions for one-mode fields defined in terms of the number and phase instead of quadratures have been studied [3, 4]. The joint definition for a two-mode field can be then handled to derive, as a marginal distribution, a Wigner function for number and phase differences. This marginal distribution establishes a correspondence between functions and operators acting on finite-dimensional spaces, providing in fact a formalism alternative to the one followed throughout this work. Its derivation and its properties are addressed in this section. First, we will consider the Wigner function introduced in [3] and afterwards a closely related definition contained in [4].

The Wigner correspondence $A \leftrightarrow W_A(\tilde{n},\varphi)$ [3] for the number $\tilde{n}$ and phase φ is expressible by means of an operator kernel $\Delta(\tilde{n},\varphi)$ in the form

$$W_A(\tilde{n},\varphi) = \mathrm{tr}[A\Delta(\tilde{n},\varphi)] \qquad A = 2\pi\sum_{\tilde{n}=0}^{\infty}\int \mathrm{d}\varphi\, W_A(\tilde{n},\varphi)\Delta(\tilde{n},\varphi). \tag{4.1}$$

In this definition the number variable $\tilde{n}$ takes integer as well as half integer values

$$\tilde{n} = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots \tag{4.2}$$

and this is the range of variation (step 1/2) of the previous sum. On the other hand, φ takes any value in a 2π interval. The operator kernel is

$$\Delta(\tilde{n},\varphi) = \frac{1}{2\pi}\int \mathrm{d}\varphi'\,\mathrm{e}^{-2\mathrm{i}\tilde{n}\varphi'}|\varphi+\varphi'\rangle\langle\varphi-\varphi'| \tag{4.3}$$

where

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}n\varphi}|n\rangle \tag{4.4}$$

are the Susskind–Glogower phase states [14].

The joint Wigner function for a two-mode field will be given by

$$\begin{aligned} &W_A(\tilde{n}_1,\tilde{n}_2,\varphi_1,\varphi_2) = \mathrm{tr}[A\Delta(\tilde{n}_1,\varphi_1)\Delta(\tilde{n}_2,\varphi_2)] \\ &A = (2\pi)^2\sum_{\tilde{n}_1,\tilde{n}_2=0}^{\infty}\int \mathrm{d}\varphi_1\,\mathrm{d}\varphi_2 W_A(\tilde{n}_1,\tilde{n}_2,\varphi_1,\varphi_2)\Delta(\tilde{n}_1,\varphi_1)\Delta(\tilde{n}_2,\varphi_2). \end{aligned} \tag{4.5}$$

Since here we are interested only in the number and phase differences, we will consider the functions

$$W_A(\tilde{n}_1,\tilde{n}_2,\phi) = \int \mathrm{d}\varphi\, W_A(\tilde{n}_1,\tilde{n}_2,\varphi,\varphi+\phi) = \mathrm{tr}[A\Delta(\tilde{n}_1,\tilde{n}_2,\phi)] \tag{4.6}$$

where

$$\Delta(\tilde{n}_1,\tilde{n}_2,\phi) = \int \mathrm{d}\varphi\Delta(\tilde{n}_1,\varphi)\Delta(\tilde{n}_2,\varphi+\phi). \tag{4.7}$$

The explicit form for the operator kernel is

$$\Delta(\tilde{n}_1,\tilde{n}_2,\phi) = \frac{1}{2\pi}\int \mathrm{d}\phi'\,\mathrm{e}^{2\mathrm{i}\tilde{m}\phi'}|N,\phi+\phi'\rangle\langle N,\phi-\phi'| \tag{4.8}$$

where the phase difference state

$$|N,\phi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{N}\mathrm{e}^{-\mathrm{i}(N/2-n)\phi}|N-n,n\rangle \tag{4.9}$$

lies in the finite-dimensional space H_N,

$$N = \tilde{n}_1 + \tilde{n}_2 \tag{4.10}$$

represents the total photon number and

$$\tilde{m} = \tfrac{1}{2}N - \tilde{n}_2 = \tfrac{1}{2}(\tilde{n}_1 - \tilde{n}_2) \tag{4.11}$$

represents half of the number difference.

We can see that $\Delta(\tilde{n}_1, \tilde{n}_2, \phi)$ commutes with the total photon number and (4.8) is an operator acting exclusively on H_N. Since N is integer, $\tilde{n}_1$ and $\tilde{n}_2$ must be both integers or half integers simultaneously. Otherwise $\Delta(\tilde{n}_1, \tilde{n}_2, \phi)$ vanish. From now on we use these parameters N and $\tilde{m}$ renaming the operator kernel and the Wigner function as

$$\Delta(\tilde{n}_1, \tilde{n}_2, \phi) = \Delta^{(N)}(\tilde{m}, \phi) \qquad W_A(\tilde{n}_1, \tilde{n}_2, \phi) = W_A^{(N)}(\tilde{m}, \phi) \tag{4.12}$$

and

$$W_A^{(N)}(\tilde{m}, \phi) = \mathrm{tr}[A\Delta^{(N)}(\tilde{m}, \phi)]. \tag{4.13}$$

Relations (4.13), (4.12) and (4.8) provide a phase-space formalism for operators acting on finite-dimensional spaces H_N of arbitrary dimension $N+1$, which can be compared with the procedure we have developed in this paper. The phase space is different. The ϕ variable can take any value within a 2π interval. Concerning $\tilde{m}$, for a given value of N we have from (4.10) that $\tilde{n}_2$ can take values $\tilde{n}_2 = 0, 1/2, 1, 3/2, \ldots, N$, and then $\tilde{m}$ can take the $2N+1$ values

$$\tilde{m} = -\tfrac{1}{2}N, -\tfrac{1}{2}N + \tfrac{1}{2}, -\tfrac{1}{2}N + 1, \ldots, \tfrac{1}{2}N - 1, \tfrac{1}{2}N - \tfrac{1}{2}, \tfrac{1}{2}N \tag{4.14}$$

which are all the integers and half integers from $-N/2$ to $N/2$. These are the ranges of variation of these quantities whenever they appear.

From now we concentrate on any H_N and on operators A and B acting exclusively on it, $A, B : H_N \to H_N$. Next we study the properties of this phase-space formalism. First, we have that (4.13) can be inverted,

$$A = 2\pi \sum_{\tilde{m}} \int \mathrm{d}\phi\, W_A^{(N)}(\tilde{m}, \phi)\Delta^{(N)}(\tilde{m}, \phi) \tag{4.15}$$

as can be seen by direct calculation. From this relation the traciality

$$\mathrm{tr}(AB) = 2\pi \sum_{\tilde{m}} \int \mathrm{d}\phi\, W_A^{(N)}(\tilde{m}, \phi) W_B^{(N)}(\tilde{m}, \phi) \tag{4.16}$$

can be proved.

This equation guarantees that if $A \neq B$ then $W_A^{(N)} \neq W_B^{(N)}$ so different operators have different Wigner functions. However, there are many $W_A^{(N)}(\tilde{m}, \phi)$ compatible with (4.15). This is because there are functions $W^{(N)}(\tilde{m}, \phi) \neq 0$ such that

$$\sum_{\tilde{m}} \int \mathrm{d}\phi\, W^{(N)}(\tilde{m}, \phi)\Delta^{(N)}(\tilde{m}, \phi) = 0 \tag{4.17}$$

which can be added to any $W_A^{(N)}(\tilde{m}, \phi)$. These functions are those satisfying

$$\int \mathrm{d}\phi\, W^{(N)}\left(\frac{m+m'}{2}, \phi\right) \mathrm{e}^{\mathrm{i}\phi(m-m')} = 0 \tag{4.18}$$

for every $m, m' = -N/2, -N/2+1, \ldots, N/2-1, N/2$.

The form (4.8) ensures that $\Delta^{(N)\dagger}(\tilde{m}, \phi) = \Delta^{(N)}(\tilde{m}, \phi)$, which is the property of reality.

Concerning marginals, for the phase difference we have

$$\sum_{\tilde{m}} \Delta^{(N)}(\tilde{m}, \phi) = |N, \phi\rangle\langle N, \phi| \tag{4.19}$$

whereas for the number difference

$$\int \mathrm{d}\phi\, \Delta^{(N)}(\tilde{m}, \phi) = \begin{cases} |N, \tilde{m}\rangle\langle N, \tilde{m}| & \text{if } \tfrac{1}{2}N \pm \tilde{m} \text{ integer} \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

where $|N, \tilde{m}\rangle = |n_1 = \tfrac{1}{2}N + \tilde{m}, n_2 = \tfrac{1}{2}N - \tilde{m}\rangle$.

It transforms properly under any phase-difference shift,

$$e^{i\theta j_z}\Delta^{(N)}(\tilde{m},\phi+\theta)\,e^{-i\theta j_z}=\Delta^{(N)}(\tilde{m},\phi) \tag{4.21}$$

where j_z represents here half of the number-difference operator. It behaves properly under the parity transformation

$$P\Delta^{(N)}(-\tilde{m},-\phi)P=\Delta^{(N)}(\tilde{m},\phi) \tag{4.22}$$

and also under conjugation

$$\begin{aligned}\langle n_1,n_2|\Delta^{(N)}(\tilde{m},-\phi)|n_1',n_2'\rangle^*&=\langle n_1,n_2|\Delta^{(N)}(\tilde{m},\phi)|n_1',n_2'\rangle\\ \langle N,\phi'|\Delta^{(N)}(-\tilde{m},\phi)|N,\phi''\rangle^*&=\langle N,\phi'|\Delta^{(N)}(\tilde{m},\phi)|N,\phi''\rangle.\end{aligned} \tag{4.23}$$

All this is valid without any difference between even and odd dimensions. In comparison with the formalism studied in the preceding sections, the approach developed here differs mainly in the underlying phase space which accounts for the differences we provide in detail in the following. Concerning the number difference, we have here an enlarged phase space ($2N+1$ values instead of $N+1$) where there are always $\tilde{m}$ values which are not in the spectrum of half of the number difference within H_N. These additional values disappear when considering the corresponding marginal distribution (4.20). With respect to the phase difference ϕ, any value in a 2π interval is allowed for every subspace H_N in front of the $N+1$ values of the preceding sections. This allows a different transformation law in (4.21) since the phase shift can take any value. On the other hand, the marginal for the phase difference does not define an operator in the usual sense (orthogonal projectors or projection measure) but a non-orthogonal positive operator measure. Concerning unitary translations of number difference and Fourier transforms, there seem to be no suitable transformation laws because of the different form of the phase space and the dissimilarity between number difference and phase difference. Finally, another consequence of this phase space is that the correspondence $A\leftrightarrow W_A^{(N)}$ is not one to one.

In relation to the continuous range of variation for ϕ, we may think that it is overcomplete in order to describe a variable in a finite-dimensional space where we would expect a discrete character. In fact, we can show that a finite number of ϕ values are enough to know the Wigner function at any other phase-difference point. This is because we have

$$|N,\phi\rangle\langle N,\phi|=\sum_{\tilde{s}}C(\tilde{s},\phi)|N,\tilde{\phi}_{\tilde{s}}\rangle\langle N,\tilde{\phi}_{\tilde{s}}| \tag{4.24}$$

where

$$C(\tilde{s},\phi)=\frac{1}{2N+1}\sum_{\ell=-N}^{N}e^{i\ell(\tilde{\phi}_{\tilde{s}}-\phi)} \tag{4.25}$$

ℓ always being integer, and

$$\tilde{\phi}_{\tilde{s}}=\frac{4\pi}{2N+1}\tilde{s}\qquad \tilde{s}=-\frac{N}{2},-\frac{N}{2}+\frac{1}{2},-\frac{N}{2}+1,\ldots,\frac{N}{2}-1,\frac{N}{2}-\frac{1}{2},\frac{N}{2} \tag{4.26}$$

so the ranges of variation of $\tilde{m}$ and $\tilde{s}$ are everywhere the same. Equation (4.24) allows us to write the operator kernel as

$$\Delta^{(N)}(\tilde{m},\phi)=\sum_{\tilde{s}}C(\tilde{s},\phi)\Delta^{(N)}(\tilde{m},\tilde{\phi}_{\tilde{s}}) \tag{4.27}$$

and similarly for the Wigner function

$$W_A^{(N)}(\tilde{m},\phi)=\sum_{\tilde{s}}C(\tilde{s},\phi)W_A^{(N)}(\tilde{m},\tilde{\phi}_{\tilde{s}}). \tag{4.28}$$

The knowledge of the Wigner function on the $2N+1$ phase-difference points $\tilde{\phi}_{\tilde{s}}$ gives its value at any other ϕ point. Let us note that the following relations hold:

$$
\begin{aligned}
A &= \frac{(2\pi)^2}{2N+1} \sum_{\tilde{m},\tilde{s}} W_A^{(N)}(\tilde{m}, \tilde{\phi}_{\tilde{s}}) \Delta^{(N)}(\tilde{m}, \tilde{\phi}_{\tilde{s}}) \\
\operatorname{tr}(AB) &= \frac{(2\pi)^2}{2N+1} \sum_{\tilde{m},\tilde{s}} W_A^{(N)}(\tilde{m}, \tilde{\phi}_{\tilde{s}}) W_B^{(N)}(\tilde{m}, \tilde{\phi}_{\tilde{s}}).
\end{aligned}
\tag{4.29}
$$

This shows an effective discreteness on ϕ the new phase space being formed by the $(2N+1)^2$ points $(\tilde{m}, \tilde{\phi}_{\tilde{s}})$. In comparison with the Wigner function studied in section 2, this corresponds to a phase space enlarged equally on both variables. This makes this effective formalism similar, although not equal, to a Wigner function defined directly on similarly enlarged phase spaces [9].

Finally, let us consider along the same lines what occurs when we adopt a different definition of the Wigner function for number and phase like that contained in [4]. Its operator kernel for a one-mode field is given by

$$
\Delta'(n,\varphi) = \Delta(n,\varphi) + \Delta(n-1/2,\varphi) = \frac{1}{2\pi} \int \mathrm{d}\varphi' \, \mathrm{e}^{-2\mathrm{i}n\varphi'} (1 + \mathrm{e}^{\mathrm{i}\varphi'}) |\varphi + \varphi'\rangle\langle\varphi - \varphi'| \tag{4.30}
$$

where $n = 0, 1, 2, \ldots$ is always an integer. The correspondence between operators and functions is also of the form (4.1), replacing $\Delta(\tilde{n},\varphi)$ by $\Delta'(n,\varphi)$ and $\tilde{n}$ by n when necessary. The main difference between these two approaches is that the last one is defined exclusively on integer values for the number variable which are the spectrum of the number operator. In principle, we could expect that this difference may affect the range of variation of the number difference by removing the $\tilde{m}$ values which are not in the spectrum of half of the number difference.

If we replace $\Delta(\tilde{n},\varphi)$ by $\Delta'(n,\varphi)$ in (4.7) we arrive at

$$
\Delta'(n_1, n_2, \phi) = \Delta(n_1, n_2, \phi) + \Delta(n_1 - 1/2, n_2 - 1/2, \phi) \tag{4.31}
$$

where the right-hand side terms are given by (4.8). Using parametrization (4.12),

$$
\Delta'(n_1, n_2, \phi) = \Delta^{(n_1+n_2)}\left(\frac{n_1 - n_2}{2}, \phi\right) + \Delta^{(n_1+n_2-1)}\left(\frac{n_1 - n_2}{2}, \phi\right). \tag{4.32}
$$

The first kernel is acting on $H_{n_1+n_2}$ while the second is acting on $H_{n_1+n_2-1}$.

Next we proceed to extract from (4.32) the operator kernels $\Delta^{(N)'}(\tilde{m},\phi)$ acting just on the finite-dimensional subspaces H_N. We have two kinds of contributions. One comes from the first term in (4.31) and (4.32) when $n_1 + n_2 = N$. In this case, for fixed N we have that n_2 can take the values $n_2 = 0, 1, \ldots, N$, so half of the number difference $\tilde{m}$ takes the $N+1$ values $\tilde{m} = -N/2, -N/2+1, \ldots, N/2-1, N/2$, which is its spectrum within H_N. The other contribution comes from the second operator kernel in (4.31) and (4.32) when $n_1 + n_2 - 1 = N$. In this case, n_2 can take the values $n_2 = 1, \ldots, N$, since it can be seen in (4.3) that $\Delta(-1/2, \varphi) = 0$. Half the number difference $\tilde{m}$ can take the N values $\tilde{m} = -N/2+1/2, -N/2+3/2, \ldots, N/2-3/2, N/2-1/2$, which are out of its spectrum within H_N.

Then we have found that for a given value of N there are $2N+1$ possible values for half of the number difference which are the same as in (4.14).

Therefore $\Delta^{(N)'}(\tilde{m},\phi) = \Delta^{(N)}(\tilde{m},\phi)$ and both formalisms provide the same Wigner function for a finite-dimensional space, the same phase space $(\tilde{m},\phi)$, and so they have the same properties, despite the fact that their starting points are different.

The origin of this coincidence with the appearance of $\tilde{m}$ values that we did not expect can be ascribed to the behaviour of the total photon number variable in this formalism. We have in (4.32) that to the value $n_1 + n_2$ there are contributions from the subspaces $H_{n_1+n_2}$ and $H_{n_1+n_2-1}$, since the corresponding operator kernel is acting on both subspaces simultaneously. Equivalently, as we have just shown, states defined within H_N give contributions on the values $n_1 + n_2 = N$ and $n_1 + n_2 = N + 1$. These last contributions disappear if we integrate the phase difference and we always recover the correct marginal distribution.

This can be illustrated by considering the most general one-photon field state with an expression in the number basis $|\psi\rangle = \alpha|1, 0\rangle + \beta|0, 1\rangle$ whose complete Wigner function is

$$W'_\psi(n_1, n_2, \varphi_1, \varphi_2) = \frac{1}{(2\pi)^2}[|\alpha|^2\delta_{n_1,1}\delta_{n_2,0} + |\beta|^2\delta_{n_1,0}\delta_{n_2,1} + \alpha^*\beta\, \mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}\delta_{n_1,1}\delta_{n_2,1} + \alpha\beta^*\, \mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)}\delta_{n_1,1}\delta_{n_2,1}]. \tag{4.33}$$

A contribution with $n_1 + n_2 = 2$ appears and half of the number difference takes the values $0, \pm 1/2$ instead of just $\pm 1/2$ which are the allowed values for a one-photon state. In fact, all the phase-difference information is conveyed by the phase-space points with $n_1 + n_2 = 2$. To some extent, the enlargement of the phase space that this formalism tries to avoid is not completely removed and effectively persists for the number difference.

Finally, let us note that there is a simple practical scheme for determining from measurement all these Wigner functions. They are the mean values of operators commuting with the total photon number. It has been shown that the whole statistics of operators commuting with the total photon number can be derived from photon-number measurements after an unbalanced eight-port homodyne detector when two of the input ports are in vacuum [15]. In particular, this allows one to extract the mean values of the operator kernels of this and the preceding section from the statistics of such a homodyne detection.

5. Conclusions

We have analysed the definition of a Wigner–Weyl correspondence for systems described by finite-dimensional Hilbert spaces. As a phase space we have considered that formed by the spectrum of two conjugate variables. Such a phase space is, therefore, finite and discrete with d^2 points if the Hilbert space has dimension d.

Although the phase space as a classical object would demand a continuous range of variation, the statistical interpretation allows us to expect marginal distributions strongly related to the probability distribution of observables. However, their spectrum is necessarily discrete and finite, in contrast with the case for position and linear momentum where any value is allowed. We can regard a discrete and finite phase space as another non-classical feature dictated by the quantum nature of the system described, as is for instance the lack of positiveness of the Wigner function. The correct classical limit is granted because the number of phase points grows with d and in the limit of high d they become a continuum for all practical purposes. Moreover, a phase space with just d^2 points embodies straightforwardly a one-to-one relation between operators and functions in both senses of the correspondence as we have shown.

We have constructed from basic principles the most general one-to-one correspondence between operators and functions compatible with its quantum statistical content and desirable properties under phase-space transformations. This possibility was known for odd dimension. Here we have found a set of properties that it fulfills. For even dimension we have shown that it is possible to guarantee basic properties using a phase space of just d^2 points. Therefore, its enlargement by inserting additional points is not necessary.

For Cartesian variables the properties we have discussed are sufficient to define the correspondence uniquely. We have found that this is not the case for finite dimension. The generality of the constructive method followed here has allowed us to find the class of Wigner–Weyl correspondences satisfying these properties. In particular, this shows that the previously introduced Wigner function for odd d is not the only possibility having such properties. We think that this result could be interesting in relation to the study of the basic properties fulfilled by Wigner functions. Also, this freedom makes room for the fulfillment of further specific requirements which might be necessary for the application of this formalism to other particular situations.

We have found that some familiar forms of expressing the Wigner–Weyl correspondence for Cartesian variables, which are valid for odd d, are not possible for even dimension. However, this fact is no intimately connected with the fulfillment of some basic properties, as the odd d case illustrates.

Among the possible conjugate variables describable by finite-dimensional spaces we have studied the number difference and phase difference for a two-mode field. We have compared our approach with the marginal distribution for these variables arising from a Wigner function for absolute numbers and phases. We have found that two different formalisms give the same correspondence. The phase space that this procedure defines is discrete and enlarged for the number difference and continuous for the phase difference. However, this last range of variation is not completely effective in the sense that having a finite number of points is enough to know the Wigner function at any other phase-difference point.

Finally, we find relevant the possibility of defining a well behaved phase-space formalism, which naturally embodies the description of variables like azimuthal angle or phase difference by operators, in accordance with the usual description of variables in quantum mechanics.

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Appendix

In this appendix we briefly list the properties required for the Wigner function.

The correspondence $A \leftrightarrow W_A$ should be a one-to-one linear map in both directions.

Mean values should be given by averages on phase space

$$\mathrm{tr}(AB) = \frac{1}{2j+1} \sum_{m,s=-j}^{j} W_A(m,s) W_B(m,s). \tag{A.1}$$

For the sake of conciseness we will refer to this property as traciality.

To Hermitian operators (in particular, density matrices) there should correspond real functions and *vice versa*

$$W^*_{A^\dagger}(m,s) = W_A(m,s). \tag{A.2}$$

Marginal distributions should give the probability distributions for j_z and ϕ

$$\frac{1}{2j+1}\sum_{s=-j}^{j} W_A(m,s) = \langle j,m|A|j,m\rangle$$
$$\frac{1}{2j+1}\sum_{m=-j}^{j} W_A(m,s) = \langle j,\phi_s|A|j,\phi_s\rangle. \tag{A.3}$$

According to the commutation relation (2.6) the combinations $E^k F^{\dagger\ell}$, for integers k and ℓ, represent displacements along the j_z and ϕ variables, becoming a discrete and finite equivalent of the displacement operator [2] of position and linear momentum. This analogy can be made more precise by adding a phase factor [10–12]

$$D(k,\ell) = \mathrm{e}^{-\mathrm{i}(\pi/(2j+1))k\ell} E^k F^{\dagger\ell}. \tag{A.4}$$

Under translations the Wigner function should transform by the translation of its arguments

$$W_{D(k,\ell)AD^\dagger(k,\ell)}([[m+k]],[[s+\ell]]) = W_A(m,s) \tag{A.5}$$

where $[[m]]$ is equal to m modulus $2j+1$ and $[[m]] \in [-j,j]$. To some extent this preciseness is not necessary for the angular part since s or ϕ_s are naturally $2j+1$ or 2π periodic, respectively. This is not the case for m, but a cyclic behaviour for this part of the transformation is dictated by the possibility of having an operator for the azimuthal angle [11].

Another useful transformation is given by the parity operator

$$P = \sum_{s=-j}^{j} |j,\phi_{-s}\rangle\langle j,\phi_s| = \sum_{m=-j}^{j} |j,-m\rangle\langle j,m| \tag{A.6}$$

which changes the sign of ϕ and j_z. From a phase-space perspective, this is an inversion or a π rotation as well so we can require

$$W_{PAP^\dagger}(-m,-s) = W_A(m,s). \tag{A.7}$$

Another transformation is based on the Fourier relation between the $|j,m\rangle$ and $|j,\phi_s\rangle$ states. The Fourier transform unitary operator

$$T = \sum_{m=-j}^{j} |j,\phi_m\rangle\langle j,m| = \sum_{m=-j}^{j} |j,-m\rangle\langle j,\phi_m| \tag{A.8}$$

transforms ϕ into j_z and j_z into $-\phi$, being a $\pi/2$ rotation on phase space, so we can impose

$$W_{TAT^\dagger}(-s,m) = W_A(m,s). \tag{A.9}$$

Finally, we can add two more properties arising from the complex conjugation. If $A^*_{j_z}$ is the operator given by the complex conjugation of A in the j_z basis, $\langle j,m'|A^*_{j_z}|j,m\rangle = \langle j,m'|A|j,m\rangle^*$, we may impose the transformation law

$$W^*_{A^*_{j_z}}(m,-s) = W_A(m,s) \tag{A.10}$$

where the conjugation of the Wigner function follows from the anti-unitarity of the transformation, so the equality $\mathrm{tr}(A^*_{j_z}B^*_{j_z}) = \mathrm{tr}(AB)^*$ holds when using (A.1). Similarly, if A^*_ϕ results from conjugation in the ϕ basis, $\langle j,\phi_{s'}|A^*_\phi|j,\phi_s\rangle = \langle j,\phi_{s'}|A|j,\phi_s\rangle^*$, we can impose the transformation

$$W^*_{A^*_\phi}(-m,s) = W_A(m,s). \tag{A.11}$$

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